INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS, VOL. 6, 395–401 (1986)

SHORT COMMUNICATION

COMPUTER EXTENDED SERIES FOR A THERMALLY DRIVEN GAS CENTRIFUGE

M. H. BERGER

Pob X, Bldg X1000, MS102, The Oak Ridge National Laboratory, Oak Ridge, Tennessee 37830, U.S.A.

SUMMARY

Linearized, multidimensional, thermally driven flow in a gas centrifuge can be approximately described in regions away from the ends by Onsager's homogeneous pancake equation.[1] Upon reformulation of the general problem, we find a new, simple and rigorous closed form, analytical solution by assuming a special separable solution and replacing the usual Ekman end cap boundary conditions with idealized impermeable, free slip boundary conditions. Then the flow may be described by an ordinary differential equation with solutions in terms of simple, classical functions. By identifying a small parameter, say ε, defining the semi-long bowl approximation, and assuming a power series expansion in ε, a sequence of asymptotic approximations to the master potential is obtained. Not surprisingly, the leading order term involves the well known 'long bowl' solution. Using the so-called 'solving' property of the 1-D pancake Green's function,[2] we determine the next higher order solution. This recursive process is carried out on the computer to find all the terms up to $O(\varepsilon^4)$.

Consequently, the solution of some complex rotating, viscous, heat conducting flow problems that normally require large mainframe computers can be better understood.

KEYWORDS Computer Extended Series Gas Centrifuge Theory MACSYMA

INTRODUCTION

As a model of the gas flow inside a real centrifuge, we consider Onsager's pancake equation with sources and sinks and compressible Ekman boundary layers at the horizontal surfaces. This is, of course, the approximation of the linearized compressible Navier–Stokes equations first derived for thermal drive and solved by Onsager[1]

$$L\tilde{\chi} = L_6\tilde{\chi} + B^2\tilde{\chi}_{yy} = \tilde{F}(x, y), \tag{1}$$

where $\tilde{\chi}$ is the master potential and

$$L_6\tilde{\chi} = [\mathrm{e}^x(\mathrm{e}^x\tilde{\chi}_{xx})_{xx}]_{xx},$$

subject to 9 boundary conditions:

$$\tilde{\chi}_x(0, y) = \tilde{\chi}_{xx}(0, y) = 0, \quad L_5\tilde{\chi}(0, y) = \frac{Re}{32A^{10}}\bar{\theta}_y(y),$$

$$\tilde{\chi}(\infty, y) = \tilde{\chi}_y(\infty, y) = \tilde{\chi}_x(\infty, y) = L_3\tilde{\chi}(\infty, y) = 0,$$

$$\tilde{\chi}_y(x, 0) = -4S^{-1/4}Re^{-1/2}A^4[\mathrm{e}^{x/2}\tilde{\chi}_x(x, 0)]_x,$$

$$\tilde{\chi}_y(x, y_0) = 4S^{-1/4}Re^{-1/2}A^4[\mathrm{e}^{x/2}\tilde{\chi}_x(x, y_0)]_x, \tag{2}$$

0271-2091/86/060395-07$05.00

Received 10 October 1985
Revised 20 November 1985

where

$$L_5\tilde{\chi} = [e^x(L_3\tilde{\chi})_x]_x,$$

and

$$L_3\tilde{\chi} = (e^x\tilde{\chi}_{xx})_x.$$

The stream function is

$$\tilde{\Psi} = -2A^2\tilde{\chi}_x.$$

The inhomogeneity due to internal sources and sinks in Onsager's equation is[3]

$$\tilde{F}(x, y) = \frac{B^2A^2}{2ReS}\int_x^{\infty}(\tilde{T}_y - 2\tilde{V}_y)\mathrm{d}x' - \frac{B^2A^2}{2ReS}[(e^x\tilde{U}_y)_x + (e^x\tilde{W})_{xx}], \tag{3}$$

where the mass source/sink term is neglected, and the tilde is introduced to designate a multivariate function. Other related analytical work in two dimensions is that of Wood and Morton.[3] They used the natural eigenfunctions which give the solutions as a doubly infinite generalized Fourier series in terms of special functions. But more importantly, they derived the inhomogeneities in (3). The rest of the 2-D work on this equation or its variants is numerical, using either finite elements[1] or finite differences[4] or the method of lines,[5] etc. Recently, after one not-so-small trick taken from the computational fluid dynamicists (i.e. the method of lines), this problem has been recast into a more tractable form. Herein we treat only the thermal wall drive. Our approach is more interesting than prior work from the viewpoint of understanding the approximate structure and physics of the solutions. It has the additional advantage of making the computations for this special set of natural drives a more or less trivial microcomputer programming exercise. That is, the resultant analytical formulae can be readily implemented on a small computer.

Van Dyke[6,7] talks about computing hundreds and hundreds (maybe even thousands) of terms of regular perturbation expansions using his FORTRAN language computer codes. His goal, I think, is to extend the radius of convergence, to infinity if possible, to describe the exact solution structure. I suppose this is what he means by computer extension of series. What we seek is a broader understanding of the dependencies of our solution rather than just the numerical coefficients of the powers of the small expansion parameter. For a better understanding of the pancake equation perhaps it makes more sense to have the first few terms in the expansion complete in all their glorious detail. This non-trivial goal is achieved with the aid of the MACSYMA symbolic manipulation code[8] for the quadratures. The idea here is that a simple formula is worth a thousand numbers. Furthermore, we are of the opinion that exact solutions to the asymptotic form of the approximate gas centrifuge equations of fluid motion make more sense than *ad hoc* constructs sometimes given for thermal drive, scoop drive and feed drive.[9] For instance, had Olander[9] assumed a half sine wave for the axial variation of the thermal drive instead of a modified parabolic curve fit, he would have correctly modelled the leading order thermal drive term. Perhaps this shows the essential difference between an 'engineering' or seat-of-the-pants approach and a 'rigorous' mathematical approach. In a companion paper we report on the corresponding results for the internal source/sinks.

TWO DIMENSIONAL ANALYSIS

Generalizing Viecelli's trick,[5] assume a 'special' separable solution (i.e. a 1-term Fourier sine series) of the form

$$\tilde{\chi}(x, y) = \chi(x) \sin (N\pi y/y_0), \qquad N = 1, 2, 3, \ldots \tag{4}$$

and replace the Ekman end cap boundary conditions with simpler impermeable free slip boundary conditions. Although Ekman layers present no extraordinary difficulties in robust numerical techniques they overly complicate pure analysis. Anyway, Ekman boundary layers are responsible for only about 10 per cent of the recirculation for a May machine in thermal drive at 700 mps.[1] It would be gladdening to be able to predict the hydrodynamics with 10 per cent error. Assuming that the thermo-fluid parameter B is arbitrary, i.e. $B \geqslant 0$,

$$L_6 \chi(x) - E^2 \chi(x) = F(x), \tag{5}$$

where $E \equiv BN\pi/y_0$. Boundary conditions are

$$\chi_x(0) = \chi_{xx}(0) = 0, \tag{6}$$

assuming that $\sin (N\pi y/y_0) \neq 0$. If $\tilde{\theta} = \theta(x) \cos (N\pi y/y_0)$, then

$$L_5 \chi(0) = \frac{Re}{32A^{10}} \bar{\theta}_y(0) \tag{7}$$

for $\sin (N\pi y/y_0) \neq 0$, where,

$$\bar{\theta}_y(0) = \frac{-N\pi}{y_0} \bar{\theta}(0). \tag{8}$$

As $x \to \infty$,

$$\chi(\infty) = \chi_x(\infty) = L_3 \chi(\infty) = 0 \tag{9}$$

for $\sin (N\pi y/y_0)$, $\cos (N\pi y/y_0) \neq 0$. At the horizontal ends we assume impermeable end caps as a convenient approximation:

$$\tilde{\chi}_{xx}(x, 0) = \tilde{\chi}_{xx}(x, y_0) = 0. \tag{10}$$

But these approximate end cap boundary conditions are satisfied exactly by construction. Unfortunately, the types and shapes of boundary conditions and internal sources and sinks are limited to sinusoidal-like axial distributions.

In the early years of the U.S. gas centrifuge program Ging[10] concluded that there are no purely periodic eigenfunctions (i.e. pure imaginary eigenvalues) for the homogeneous, linearized gas centrifuge flow equations with curvature. Presumably this means that any unsustained disturbance in the centrifuge decays out due to viscosity. But herein we are solely concerned with sustained harmonic disturbances on the boundary, and it has just been shown that a high speed gas centrifuge driven by a pure harmonic responds in a purely harmonic fashion.

PERTURBATION ANALYSIS: SEMI-LONG BOWL APPROXIMATION

Construction of general and particular solutions for the derived inhomogeneous o.d.e. system appears to be fraught with difficulty. Instead, we treat the simpler perturbation problem. This is possible due to the anisotropic nature of our governing partial differential equation. Define $\varepsilon = E = BN\pi/y_0$, such that $0 \leqslant \varepsilon < 1$ (which clearly limits N). Small ε is achieved whether $B \to 0$ or $y_0 \to \infty$ (i.e. short decay length ratios and/or long machines). Now, we have a differential equation with a small parameter:

$$L_6 \chi(x) - \varepsilon^2 \chi(x) = F(x). \tag{11}$$

Assuming an even-ordered power series expansion in the small positive parameter ε, say

$$\chi(x) = \chi_0(x) + \varepsilon^2 \chi_2(x) + \cdots, \tag{12}$$

and substituting into the o.d.e. we obtain

$$L_6 \chi_0 + \varepsilon^2 L_6 \chi_2 + \cdots - \varepsilon^2 \chi_0 - \varepsilon^4 \chi_2 + \cdots = F(x). \tag{13}$$

We call such an approximation the semi-long bowl solution since it derives from the long bowl solution. The large parameter limit, $E \gg 1$, which is called the short bowl solution, will be discussed separately.

Collecting like ordered terms in ε in the usual manner, with $F(x) = 0$, and postulating the general $O(\varepsilon^M)$ problem gives:

$$L_6 \chi_M = \chi_{M-2}, \qquad M = 0, 2, 4, 6, \ldots, \qquad \chi_{-2} \equiv 0,$$

$$\chi'_M(0) = \chi''_M(0) = L_5 \chi_M(0) = 0, \qquad \text{with} \qquad L_5 \chi_0(0) = \frac{Re}{32A^{10}} \bar{\theta}_y(0),$$

$$\chi_M(\infty) = \chi'_M(\infty) = L_3 \chi_M(\infty) = 0. \tag{14}$$

All that we need to solve the $O(1)$–$O(\varepsilon^M)$ problems is the Green's function. Recall[2]

$$G(x;x^*) = -\tfrac{1}{2}(\mathrm{e}^{-2x}\mathrm{e}^{-x^*} + \mathrm{e}^{-x}\mathrm{e}^{-2x^*}) + \tfrac{1}{8}\mathrm{e}^{-2x}\mathrm{e}^{-2x^*} - x\mathrm{e}^{-x}\mathrm{e}^{-x^*} + \tfrac{1}{4}(x^* - x + 3)\mathrm{e}^{-2^*}, \quad x < x^*,$$

$$G(x;x^*) = -\tfrac{1}{2}(\mathrm{e}^{-2x}\mathrm{e}^{-x^*} + \mathrm{e}^{-x}\mathrm{e}^{-2x^*}) + \tfrac{1}{8}\mathrm{e}^{-2x}\mathrm{e}^{-2x^*} - x^*\mathrm{e}^{-x}\mathrm{e}^{-x^*} + \tfrac{1}{4}(x - x^* + 3)\mathrm{e}^{-2x}, \quad x > x^*. \tag{15}$$

THE $O(\varepsilon^4)$ SOLUTION

The leading order solution (i.e. basic solution) is just the long bowl result which is repeated here for reference:

$$\chi_0(\xi) = \frac{L_5 \chi(0)}{4}(-2\mathrm{e}^{-\xi} + \tfrac{3}{2}\mathrm{e}^{-2\xi} + \xi \mathrm{e}^{-2\xi}), \qquad 0 \leqslant \xi \leqslant \infty. \tag{16}$$

Using the 'solving' property gives

$$\chi_2(x) = \int_0^\infty G(x;\xi)\chi_0(\xi)\mathrm{d}\xi = \left[\int_0^x + \int_x^\infty\right] G(x;\xi)\chi_0(\xi)\mathrm{d}\xi$$

$$= \int_0^x G_< \chi_0 + \int_x^\infty G_> \chi_0. \tag{17}$$

Specifically,

$$\chi_2(x) = \frac{L_5 \chi(0)}{13{,}824}(1292\mathrm{e}^{-x} - (864x + 831)\mathrm{e}^{-2x} - 192\mathrm{e}^{-3x} + (6x + 22)\mathrm{e}^{-4x}). \tag{18}$$

Comparing $O(1)$ terms with comparable $O(\varepsilon^2)$ terms in χ we notice that the numerical coefficients of the $O(\varepsilon^2)$ terms are smaller than the $O(1)$ terms. This suggests that for pure thermal drive our series may have a radius of convergence greater than unity. The $O(\varepsilon^4)$ corrections are computed by repeating this integration process but the complete expression is given for χ_{xx} only:

$$\chi_{xx}=\frac{-L_5\chi(0)}{82{,}944{,}000}[(1{,}336{,}906\varepsilon^4-7{,}752{,}000\varepsilon^2+41{,}472{,}000)e^{-x}+(-3{,}614{,}250\varepsilon^4 x$$
$$+20{,}736{,}000\varepsilon^2 x-82{,}944{,}000x+215{,}005\varepsilon^4-792{,}000\varepsilon^2-41{,}472{,}000)e^{-2x}$$
$$+(-1{,}938{,}000\varepsilon^4+10{,}368{,}000\varepsilon^2)e^{-3x}+(144{,}000\varepsilon^4 x-576{,}000\varepsilon^2 x+378{,}500\varepsilon^4$$
$$-1{,}824{,}000\varepsilon^2)e^{-4x}+8000\varepsilon^4 e^{-5x}-(90\varepsilon^4 x+411\varepsilon^4)e^{-6x}]. \qquad (19)$$

Generally,

$$\chi_M(x)=\int_0^\infty G(x,\xi)\chi_{M-2}(\xi)\mathrm{d}\xi, \qquad M=2,4,\ldots \qquad (20)$$

Thus a regular expansion can be derived to arbitrary order M, in theory, 'simply' by one-dimensional quadratures. Here we are limited only by the radius of convergence of our asymptotic series. This problem is perfectly well suited to Van Dyke's technique.[7]

ILLUSTRATIONS

End effects are expected to produce axial flow taper as well as reduce the peak axial flow and simultaneously produce non-zero radial flow. The flow blockage behaviour associated with impermeable end caps is obvious. Such two-dimensional effects are not depicted here in the axial mass velocity profile for thermal drive plotted by MACSYMA[8] (Figure 1 for $L_5\chi(0)=-1$ and

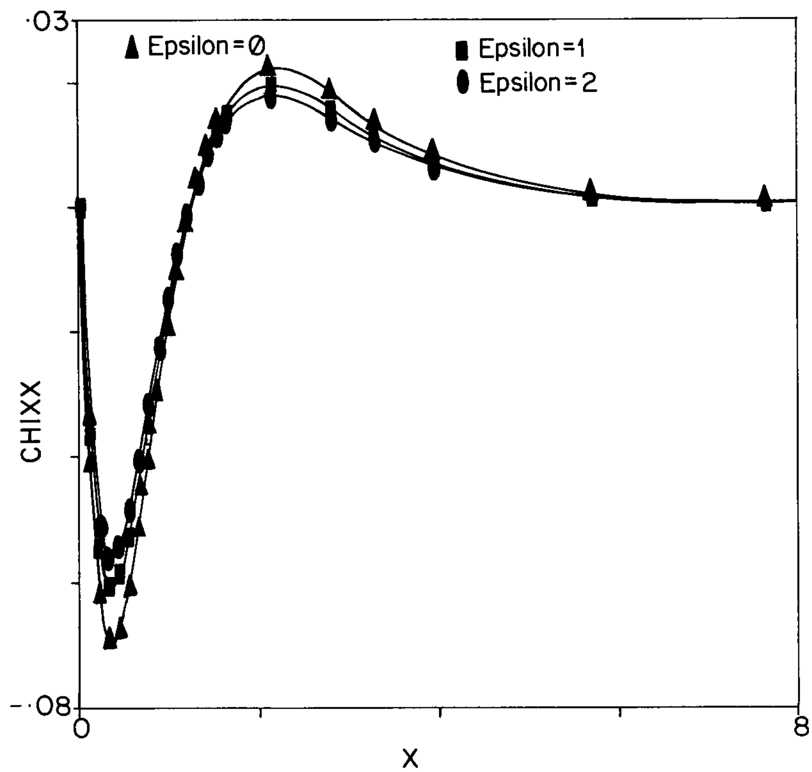

Figure 1. Axial mass velocity profile for pure thermal drive, $\rho_0 w/(4A^4)$, with $\varepsilon=0,1,2$

$\varepsilon = 0, 1, 2$). Comparing $\varepsilon = 0$ with the $O(\varepsilon^2)$ theory (not shown), we find about 3 per cent difference for $\varepsilon = 1$ and a whopping 50 per cent for $\varepsilon = 2$. The $O(\varepsilon^2)$ theory is obviously invalid for $\varepsilon \geqslant 2$ and the $O(\varepsilon^4)$ theory appears not to have broken down. The best way to calculate the flow for $\varepsilon \gg 1$ is not using this semi-long bowl theory but rather using the short bowl theory, resulting in an entirely different set of equations. It is probably expecting too much for these two different regimes of ε to overlap, resulting in a uniformly valid perturbation solution.

ACKNOWLEDGEMENT

Based on work performed at Oak Ridge Gaseous Diffusion Plant, operated by Martin Marietta Energy Systems, Inc, for the U.S. Department of Energy under U.S. Government Contract DE-AC05-84OR21400.

NOMENCLATURE

A	Stratification parameter $\left[\frac{MV_w^2}{2RT_0}\right]^{1/2}$
B	$\frac{ReS^{1/2}}{4A^6}$
E	$BN\pi/y_0$
$\tilde{F}(x, y), F(x)$	Inhomogeneities
$L\tilde{\chi}$	$L_6\tilde{\chi} + B^2\tilde{\chi}_{yy}$
$L_6\tilde{\chi}$	$[e^x(e^x\tilde{\chi}_{xx})_{xx}]_{xx}$
M	Molecular weight, and order of expansion
N	Number of half-cycles
Pr	Prandtl number
R	Universal gas constant
Re	Reynolds number
S	$1 + \frac{(\gamma - 1)}{2\gamma} PrA^2$
T_0	Reference temperature
V_w	Wall velocity
x	Scale heights variable
y	Dimensionless axial co-ordinate
y_0	Dimensionless rotor length
ε	Small expansion parameter, $BN\pi/y_0$
$\bar{\theta}(0)$	Amplitude of temperature variation
$\bar{\theta}_y$	Sidewall temperature gradient, $-N\pi\bar{\theta}(0)/y_0$
$\rho_0 w$	Dimensionless axial velocity
$\tilde{\chi}, \chi$	Master potential
$\tilde{\Psi}$	Stream function
$\sim$	Multivariate function

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